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Gel'fand pair criteria for compact matrix quantum groups

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ABSTRACT

In this paper we will give Gel'fand-type criteria for compact matrix quantum groups. Firstly we consider biinvariance in compact matrix quantum groups with respect to a quantum subgroup, and secondly an 'infinitesimal' version where we consider biinvariance with respect to certain two-sided coideals in the dual of the Hopf $*$-algebra.

0. INTRODUCTION

In his original paper of 1950, Gel'fand [G] introduced the notion of spherical functions on Riemannian symmetric spaces $X = K\backslash G$ in which G is a connected Lie group and K a compact subgroup of G. These spherical functions play an important role in the study of the involutive Banach algebra $L^1(K\backslash G/K)$ of integrable K-biinvariant functions on G, which is an algebra under convolution. In case this algebra is commutative, its continuous characters are given exactly by the bounded spherical functions on G.

In his paper Gel'fand also gave a sufficient condition for the algebra $L^1(K\backslash G/K)$ to be commutative, namely the existence of an involutive group automorphism of G which fixes the elements of K. Later this was generalized to pairs (G, K) of a locally compact group G, not necessarily of Lie type, and a compact subgroup K of G. In this setting it suffices if there exists an involutive group automorphism of G which sends the double coset of x to the double coset of x^{-1} $(x \in G)$. Such pairs (G, K), for which the algebra $L^1(K\backslash G/K)$ is commutative, are called Gel'fand pairs (see also [F] and [Go]). In particular, saying that a pair (G, K) of compact groups, in which K is a subgroup of G, is a Gel'fand pair, comes down to saying that for each irreducible unitary representation of G there is an at most one-dimensional subspace of K-biinvariant elements in the span of its matrix coefficients.

In [K3] Koornwinder introduced the notion of a compact quantum Gel'fand pair. This is a pair of compact matrix quantum groups $(\mathcal{A}, \mathcal{B})$ in which $\mathcal{B}$ is a quantum subgroup of $\mathcal{A}$, and such that for every irreducible unitary matrix corepresentation of $\mathcal{A}$ the restriction to $\mathcal{B}$ contains the trivial corepresentation of $\mathcal{B}$ with multiplicity at most one. Equivalently one can say that for each irreducible unitary matrix corepresentation of $\mathcal{A}$ the linear span in $\mathcal{A}$ of its matrix coefficients contains an at most one-dimensional subspace of biinvariant elements with respect to $\mathcal{B}$. If this subspace has dimension one, one can look at the, up to constants, unique $\mathcal{B}$-biinvariant element. These so-called spherical elements have in some cases been identified with certain q-hypergeometric functions. The first to do so were Vaksman and Soibelman in their paper [VS] (see also [M] and [K1]). Later Noumi, Yamada and Mimachi [NYM] recovered the spherical functions on the quantum $(2n-1)$-sphere $U_q(n-1)\backslash U_q(n)$ as little q-Jacobi polynomials.

A slightly more general point of view was first introduced by Koornwinder in [K2] (see also [K4]). He looked at biinvariant elements in the quantized function algebra, not with respect to a quantum subgroup, but with respect to certain two-sided coideals in the corresponding quantized universal enveloping algebra. These 'infinitesimally' biinvariant elements were shown to be expressible in terms of q-Legendre and Askey–Wilson polynomials. Also, a few years ago Ueno and Takebayashi [UT] stated that they had been able to identify the zonal spherical functions on the quantum homogeneous space $GL_q(n)/O_q(n)$, for $n = 3$, with Macdonald's q-polynomials associated to root system A_n. Noumi [N1] announced this result for general n, and recently has given detailed proofs of it in [N2]. This result incorporates Koornwinder's [K4]. A study, with extensive examples, of Gel'fand pairs of quantum groups, related q-special functions and their connection with hypergroups can be found in [V].

The purpose of this paper is to present two criteria for the above-mentioned Gel'fand property, analogous to the criterion in the classical case. In the first section we recall some facts on compact matrix quantum groups, and we deal with the quantum subgroup case. In section two we will consider the infinitesimal case. In both cases our result will be illustrated by two examples.

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1. THE SUBGROUP CASE

Compact matrix quantum groups were introduced by Woronowicz [W]. In treating their representation theory he showed the existence of a Haar functional and proved the quantum analogue of Schur's orthogonality relations and the Peter–Weyl theorem. Woronowicz' approach relies heavily on the use of C^*-algebras. Koornwinder [K5] gives a more simple definition of compact matrix quantum groups and a shorter derivation of the main results of [W], avoiding C^*-algebras (see also [D]). We briefly summarize the main results of [K5].

A *Hopf $*$-algebra* is a Hopf algebra $\mathcal{A}$ over $\mathbb{C}$, equipped with an involution $*: \mathcal{A} \to \mathcal{A}$, such that as a unital algebra $\mathcal{A}$ becomes a unital $*$-algebra, and such that the comultiplication $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and the counit $\varepsilon: \mathcal{A} \to \mathbb{C}$ are both $*$-homomorphisms. Then it can be shown that the antipode S is invertible and satisfies $S \circ * \circ S \circ * = id$.

A matrix corepresentation $t = (t_{ij})_{1 \leq i,j \leq n}$ of $\mathcal{A}$ is a square matrix with entries in $\mathcal{A}$, such that $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$ and $\varepsilon(t_{ij}) = \delta_{ij}$. t is called *unitary* if $S(t_{ij}) = t_{ji}^*$ for all $1 \leq i,j \leq n$.

If $\mathcal{A}$ is a Hopf $*$-algebra, then we call $\mathcal{A}$ *associated with a compact matrix quantum group* if there is a finite dimensional corepresentation $(u_{ij})_{1 \leq i,j \leq n}$ of $\mathcal{A}$ such that both (u_{ij}) and (u_{ij}^*) are unitarizable (i.e. are equivalent to unitary matrix corepresentations of $\mathcal{A}$) and if $\mathcal{A}$ as an algebra is generated by the u_{ij} and u_{ij}^* $(i,j = 1, \ldots, n)$. Equivalently: if $\mathcal{A}$ is generated by the matrix elements of one of its unitary matrix corepresentations.

In [K5] Koornwinder proves the following:

Let $\mathcal{A}$ be the Hopf $*$-algebra associated with a compact matrix quantum group, and let $\{(t_{ij}^\alpha)\}_{\alpha \in \hat{\mathcal{A}}}$ be a maximal set of inequivalent irreducible unitary matrix corepresentations of $\mathcal{A}$. Then:

(1) the t_{ij}^α form a linear basis for $\mathcal{A}$

(2) if we define the linear functional $h: \mathcal{A} \to \mathbb{C}$ by $h(t_{ij}^\alpha) = \delta_{\alpha 1}$ ($\alpha = 1$ corresponds to the trivial corepresentation given by the one by one matrix $(1_{\mathcal{A}})$), then h satisfies the following three properties:

(i) $h(1_{\mathcal{A}}) = 1$

(ii) $(h \otimes id)\Delta(a) = h(a)1_{\mathcal{A}} = (id \otimes h)\Delta(a) \quad$ for all $a \in \mathcal{A}$

(iii) $h(aa^*) > 0 \quad$ if $a \neq 0$

and as a linear functional, h is uniquely determined, up to normalization, by the first (or second) equality in (ii). h is called the *Haar functional* on $\mathcal{A}$.

Let $\mathcal{B}$ be the Hopf $*$-algebra associated with another compact matrix quantum group, such that there exists a surjective Hopf $*$-algebra homomorphism $\psi: \mathcal{A} \to \mathcal{B}$. We call $\mathcal{B}$ a *quantum subgroup* of $\mathcal{A}$.

Definition. We say that $a \in \mathcal{A}$ is *$\mathcal{B}$-biinvariant* if

$$(\psi \otimes id)\Delta(a) = 1_{\mathcal{B}} \otimes a$$

$$(id \otimes \psi)\Delta(a) = a \otimes 1_{\mathcal{B}}$$

where $1_{\mathcal{B}}$ denotes the unit element of $\mathcal{B}$. Write ${}_{\mathcal{B}}\mathcal{A}_{\mathcal{B}}$ for the $*$-subalgebra of $\mathcal{B}$-biinvariant elements in $\mathcal{A}$.

Definition. The pair $(\mathcal{A}, \mathcal{B})$ is called a *compact quantum Gel'fand pair* if the dimension of $\mathcal{B}$-biinvariant elements in $\mathrm{span}\{t_{ij}^{\alpha}\}$ is ≤ 1 for each irreducible unitary matrix corepresentation (t_{ij}^{α}) of $\mathcal{A}$ ('span' will always denote the linear span over $\mathbb{C}$).

Equivalently: $\dim\{c \in \mathbb{C}^n : (\psi(t_{ij}^{\alpha}))c = c1_{\mathcal{B}}\} \leq 1$ for all irreducible unitary matrix corepresentations (t_{ij}^{α}) of $\mathcal{A}$.

In other words, we demand that for each irreducible unitary matrix corepresentation of $\mathcal{A}$ its restriction to $\mathcal{B}$ via ψ contains the trivial corepresentation $(1_{\mathcal{B}})$ of $\mathcal{B}$ with multiplicity at most one.

This definition was first given in [K3] (see also the remark at the end of this paper).

Lemma 1.1. *Suppose $\sigma_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ is an injective, (anti-)comultiplicative linear mapping. Then $h_{\mathcal{B}} \circ \sigma_{\mathcal{B}} = ch_{\mathcal{B}}$, if $h_{\mathcal{B}}$ denotes the normalized Haar functional on $\mathcal{B}$. The constant c is given by $c = h_{\mathcal{B}}(\sigma_{\mathcal{B}}(1_{\mathcal{B}}))$.*

Proof. Verify that $h_{\mathcal{B}} \circ \sigma_{\mathcal{B}}$ satisfies the condition (ii) that determines $h_{\mathcal{B}}$ up to constants. □

We can now state the following result:

Theorem 1.2. *Let $\mathcal{A}, \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{B}$ be as before. Suppose furthermore that there exist a bijective, anti-comultiplicative linear map*

$$\sigma : \mathcal{A} \to \mathcal{A}$$

and an injective, anti-comultiplicative linear map

$$\sigma_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$$

such that

(i) *the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sigma} & \mathcal{A} \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathcal{B} & \xrightarrow{\sigma_{\mathcal{B}}} & \mathcal{B} \end{array}$$

commutes

(ii) $\sigma(a) = a$ *for all* $a \in {}_{\mathcal{B}}\mathcal{A}_{\mathcal{B}}$.

Then the pair $(\mathcal{A}, \mathcal{B})$ is a compact quantum Gel'fand pair.

Proof. Let $\{(t_{ij}^{\alpha})\}_{\alpha \in \hat{\mathcal{A}}}$ be a maximal set of inequivalent irreducible unitary matrix corepresentations of $\mathcal{A}$. Pick an element $(t_{ij}^{\alpha})_{1 \leq i,j \leq n}$ of this set and suppose that

$$\dim\{c \in \mathbb{C}^n : (\psi(t_{ij}^{\alpha}))c = c1_{\mathcal{B}}\} = m$$

with $m \geq 1$. Possibly after a change of basis in the corresponding corepresentation space, we may assume that

$$
(\psi(t_{ij}^{\alpha})) = \begin{pmatrix} 1_{\mathcal{B}} & & & & 0 \\ & \ddots & & & \\ & & 1_{\mathcal{B}} & & \\ & & & \boxed{*} & \\ & & & & \ddots \\ 0 & & & & & \boxed{*} \end{pmatrix}
$$

where $1_{\mathcal{B}}$ appears m times and the submatrices marked by a star correspond to irreducible unitary corepresentations of $\mathcal{B}$, inequivalent to $(1_{\mathcal{B}})$. Then

$$
\operatorname{span}\{t_{ij}^{\alpha}\}_{i,j=1}^{n} \cap {}_{\mathcal{B}}\mathcal{A}_{\mathcal{B}} = \operatorname{span}\{t_{ij}^{\alpha}\}_{i,j=1}^{m}.
$$

If we write $h_{\mathcal{B}}$ for the normalized Haar functional on $\mathcal{B}$, this implies that $(h_{\mathcal{B}} \circ \psi)(t_{ij}^{\alpha}) \neq 0$ iff $1 \leq i = j \leq m$, and in that case $(h_{\mathcal{B}} \circ \psi)(t_{ij}^{\alpha}) = h_{\mathcal{B}}(1_{\mathcal{B}}) = 1$ $(1 \leq i \leq m)$.
Use this in the right-hand side of the equality

$$
\begin{aligned}
&(id \otimes (h_{\mathcal{B}} \circ \psi) \otimes id)(\Delta \otimes id)\Delta\sigma(t_{ij}^{\alpha}) \\
&\quad = (id \otimes (h_{\mathcal{B}} \circ \psi) \otimes id)(\Delta \otimes id)\Delta(t_{ij}^{\alpha}),
\end{aligned}
$$

which holds whenever $1 \leq i, j \leq m$. Then this right-hand side becomes

$$
\sum_{k,l=1}^{n} (id \otimes (h_{\mathcal{B}} \circ \psi) \otimes id)(t_{ik}^{\alpha} \otimes t_{kl}^{\alpha} \otimes t_{lj}^{\alpha}) = \sum_{k=1}^{m} t_{ik}^{\alpha} \otimes t_{kj}^{\alpha}.
$$

For the left-hand side use that

$$
(\Delta \otimes id)\Delta\sigma = (\sigma \otimes \sigma \otimes \sigma)(\tau\Delta \otimes id)\tau\Delta,
$$

where $\tau : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ denotes the flip homomorphism $\tau(a \otimes b) = b \otimes a$. This gives that the left-hand side equals

$$
\begin{aligned}
&\sum_{k,l=1}^{n} (id \otimes (h_{\mathcal{B}} \circ \psi) \otimes id)(\sigma \otimes \sigma \otimes \sigma)(t_{lj}^{\alpha} \otimes t_{kl}^{\alpha} \otimes t_{ik}^{\alpha}) \\
&\quad = \sum_{k,l=1}^{n} (h_{\mathcal{B}} \circ \psi \circ \sigma)(t_{kl}^{\alpha})\sigma(t_{lj}^{\alpha}) \otimes \sigma(t_{ik}^{\alpha}) \\
&\quad = \sum_{k,l=1}^{n} (h_{\mathcal{B}} \circ \sigma_{\mathcal{B}} \circ \psi)(t_{kl}^{\alpha})\sigma(t_{lj}^{\alpha}) \otimes \sigma(t_{ik}^{\alpha}) \\
&\quad = c \sum_{k=1}^{m} t_{kj}^{\alpha} \otimes t_{ik}^{\alpha}.
\end{aligned}
$$

Here we used Lemma 1.1. Actually, $c = 1$ since (ii) yields that $\sigma(1_{\mathcal{A}}) = 1_{\mathcal{A}}$ and hence by (i) we see that $\sigma_{\mathcal{B}}(1_{\mathcal{B}}) = 1_{\mathcal{B}}$. So we find

$$
\sum_{k=1}^{m} t_{ik}^{\alpha} \otimes t_{kj}^{\alpha} = \sum_{k=1}^{m} t_{kj}^{\alpha} \otimes t_{ik}^{\alpha} \quad (1 \leq i, j \leq m).
$$

Now use the linear independence of the elements $\{t_{rs}^{\alpha}\}$, and hence of the elements $\{t_{rs}^{\alpha} \otimes t_{uv}^{\alpha}\}$, to conclude that $m = 1$. $\square$

Let us give two examples to illustrate this result.

Example 1. (cf. [K3]). Put $\mathcal{A} = \mathrm{Pol}(SU_q(2))$ and $\mathcal{B} = \mathrm{Pol}(S_q(U(1) \times U(1))) =$ Hopf $*$-algebra generated by z, z^{-1} subject to the relations $zz^{-1} = 1 = z^{-1}z$, and with Hopf $*$-structure $\Delta(z^{\pm 1}) = z^{\pm 1} \otimes z^{\pm 1}$, $\varepsilon(z^{\pm 1}) = 1$, $S(z^{\pm 1}) = z^{\mp 1}$ and $(z^{\pm 1})^* = z^{\mp 1}$.
Define the surjective Hopf $*$-algebra homomorphism

$$\psi : \mathcal{A} \to \mathcal{B}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

and the bijective, multiplicative and anti-comultiplicative mappings

$$\sigma : \mathcal{A} \to \mathcal{A}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

and

$$\sigma_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}.$$

These satisfy the conditions of the theorem, since ${}_{\mathcal{B}}\mathcal{A}_{\mathcal{B}} = \mathbb{C}[\beta\gamma]$. Hence $(\mathcal{A}, \mathcal{B})$ is a compact quantum Gel'fand pair.

Example 2. (cf. [NYM]). Let $\mathcal{A} = A(U_q(n))$, with generators x_{ij} $(1 \leq i, j \leq n)$, $det_q\, X^{-1}$ and $\mathcal{B} = A(U_q(n-1))$, with generators y_{ij} $(1 \leq i, j \leq n-1)$, $det_q\, Y^{-1}$.
For the Hopf $*$-structure we refer to [NYM] (Sections 1.1, 1.3, 3.1).
Define the surjective Hopf $*$-algebra homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ by

$$\psi(x_{ij}) = y_{ij} \quad (1 \leq i, j \leq n-1)$$

$$\psi(x_{in}) = 0 = \psi(x_{ni}) \quad (1 \leq i \leq n-1)$$

$$\psi(x_{nn}) = 1$$

$$\psi(det_q\, X^{-1}) = det_q\, Y^{-1}.$$

Furthermore, define

$$\sigma : \mathcal{A} \to \mathcal{A}$$

$$x_{ij} \mapsto x_{ji}$$

$$det_q\, X^{-1} \mapsto det_q\, X^{-1}$$

and

$$\sigma_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$$

$$y_{ij} \mapsto y_{ji}$$

$$det_q\, Y^{-1} \mapsto det_q\, Y^{-1},$$

extended as algebra homomorphisms. Then σ and $\sigma_{\mathcal{B}}$ are bijective, multiplicative and anti-comultiplicative linear mappings such that $\psi \circ \sigma = \sigma_{\mathcal{B}} \circ \psi$.

Moreover, since ${}_{\mathcal{B}}\mathcal{A}_{\mathcal{B}}$ is the $*$-subalgebra of $\mathcal{A}$ generated by x_{nn} and x_{nn}^* (from [NYM], Proposition 4.4), and $\sigma(x_{nn}) = x_{nn}$, $\sigma(x_{nn}^*) = x_{nn}^*$, we conclude that $(A(U_q(n)), A(U_q(n-1)))$ is a compact quantum Gel'fand pair.

2. THE INFINITESIMAL CASE

Let us now look at a slightly more general situation, namely the case where we do not have a quantum subgroup at our disposal.

Suppose again that $\mathcal{A}$ is the Hopf $*$-algebra associated with a compact matrix quantum group. Suppose further that $\mathcal{A}$ is in duality with a Hopf $*$-algebra $\mathcal{U}$. This means that there exists a bilinear form $\langle .,. \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ such that for all $X, Y \in \mathcal{U}$, $a, b \in \mathcal{A}$ we have:

$$
\begin{aligned}
\langle XY, a\rangle &= \langle X \otimes Y, \Delta(a)\rangle & \qquad \langle X, ab\rangle &= \langle \Delta X, a \otimes b\rangle \\
\langle X, 1_{\mathcal{A}}\rangle &= \varepsilon(X) & \qquad \langle 1_{\mathcal{U}}, a\rangle &= \varepsilon(a) \\
\langle X, S(a)\rangle &= \langle S(X), a\rangle & & \\
\langle X, a^*\rangle &= \overline{\langle S(X)^*, a\rangle} & &
\end{aligned}
$$

where we have denoted the Hopf algebra operations on the two different algebras in the same way. The bar denotes complex conjugation.
We then have natural left and right actions of $\mathcal{U}$ on $\mathcal{A}$, given by

$$
\begin{aligned}
X.a &= (id \otimes X)\Delta(a) \\
a.X &= (X \otimes id)\Delta(a).
\end{aligned}
$$

Here the element X in the first (second) equality should be paired with the second (first) factor of $\Delta(a)$.

Recall that a two-sided coideal in a coalgebra $\mathcal{U}$ is defined as a linear subspace J of $\mathcal{U}$ such that $\Delta(J) \subset \mathcal{U} \otimes J + J \otimes \mathcal{U}$ and $\varepsilon(J) = 0$. For a given two-sided coideal J in $\mathcal{U}$, we define the collection of infinitesimally J-biinvariant elements in $\mathcal{A}$:

$$ {}_J\mathcal{A}_J = \{a \in \mathcal{A} : a.J = 0 = J.a\}. $$

This is a subalgebra of $\mathcal{A}$.

From now on we assume that J is $*$-invariant, i.e. $J^* = J$.

Definition. A *representation* of J on a finite dimensional Hilbert space V is a linear mapping $\pi : J \to \mathrm{End}(V)$ of J into the linear space of endomorphisms of V.

V, or (π, V), is called a *J-module*. If in addition $\pi(X^*) = \pi(X)^*$ for all $X \in J$, we call π a *$*$-representation* of J.

A linear subspace W of V is called *J-invariant* if $\pi(X)W \subset W$ for all $X \in J$, and π is called *irreducible* if the only J-invariant subspaces of V are $\{0\}$ and V itself. By the fact that J is equipped with a $*$-operation, it immediately follows that every $*$-representation of J is completely reducible, i.e. decomposes as a direct sum of irreducible $*$-representations. Furthermore, two representations (π, V) and (ρ, W) of J are said to be *equivalent* if there exists an invertible linear operator $T : V \to W$ such that $T \circ \pi(X) = \rho(X) \circ T$ for all X in J.

If π is a representation of J on a vector space V with inner product $(.,.)$, and if $\pi_{ij}(X) = (\pi(X)e_j, e_i)$ are the matrix elements of π with respect to some orthonormal basis $\{e_k\}$ of V, then write ${}^t\pi$ for the representation of J with matrix elements ${}^t\pi_{ij}(X) = \pi_{ji}(X)$.

Denote by $\bar{J}$ the algebra generated by the elements of J. For arbitrary $a \in \mathcal{A}$ put

$$ {}_aV = \operatorname{span}\{X.a : X \in \bar{J}\} $$

$$ V_a = \operatorname{span}\{a.X : X \in \bar{J}\}. $$

Then ${}_aV$ and V_a are both finite dimensional vector spaces. They become J-modules under the J-actions

$$ L_a : J \times {}_aV \to {}_aV $$

$$ (X, b) \mapsto L_a(X)b = X.b $$

and

$$ R_a : V_a \times J \to V_a $$

$$ (b, X) \mapsto R_a(X)b = b.X $$

respectively.

Definition. Let (π^1, V^1) and (π^2, V^2) be two irreducible J-modules. We call $a \in \mathcal{A}$, $a \neq 0$, of *left J-type* π^1 if ${}_aV = mV^1$ (as a direct sum of J-modules) for some $m \in \mathbb{N}$ and of *right J-type* π^2 if $V_a = nV^2$ for some $n \in \mathbb{N}$.

a is called *of double J-type* (π^1, π^2) if a is of left J-type π^1 and of right J-type π^2, or in other words, $L_a = m\pi^1$ and $R_a = n\pi^2$ for certain $m, n \in \mathbb{N}$.

Lemma 2.1. *Each $a \in \mathcal{A}$ has a unique decomposition*

$$ a = \sum_{\substack{(\pi^1, \pi^2) \\ \text{finite}}} a_{\pi^1, \pi^2} $$

in which a_{π^1, π^2} is of double J-type (π^1, π^2). Moreover, the π^1, π^2 are equivalent to $*$-representations of J.

Proof. Let $(t_{ij}^{\alpha})_{1 \leq i,j \leq d_\alpha}$ $(\alpha \in \hat{\mathcal{A}})$ be an irreducible unitary matrix corepresentation of $\mathcal{A}$ on a Hilbert space V^α of finite dimension d_α. Let $(\pi_{ij}^{\alpha})_{i \leq i,j \leq d_\alpha}$ denote the corresponding matrix representation of $\mathcal{U}$. They are related by the formula

$$ \pi^\alpha(X)v = (id \otimes X)t^\alpha(v) \quad (v \in V^\alpha) $$

where $t^\alpha : V^\alpha \to V^\alpha \otimes \mathcal{A}$ is the right coaction of $\mathcal{A}$ on V^α.

Since $J^* = J$, the $*$-representation π^α is completely reducible when restricted to J. Possibly after an orthogonal change of basis in the representation space V^α, we may assume that $\pi^\alpha(X)$ $(X \in J)$ has a block diagonal form in which each block corresponds to an irreducible $*$-representation of J. Then for given $j \in \{1, \ldots, d_\alpha\}$ there are positive integers $l_j \leq j \leq r_j$ and an irreducible $*$-representation $\pi^{\alpha(j)}$ of J such that $\pi^{\alpha(j)}$ appears in the decomposition of π^α as the block with row and column indices $l_j, l_j + 1, \ldots, r_j$.

Now fix $i, j \in \{1, \ldots, d_\alpha\}$. From the fact that for each $X \in \mathcal{U}$ we have

$$\begin{aligned} X.t^\alpha_{ij} &= \sum_{k=1}^{d_\alpha} \langle X, t^\alpha_{kj} \rangle t^\alpha_{ik} \\ &= \sum_{k=1}^{d_\alpha} \pi^\alpha_{kj}(X)\, t^\alpha_{ik}, \end{aligned}$$

it follows that ${}_{t^\alpha_{ij}}V$ is spanned by the t^α_{ik} with $l_j \leq k \leq r_j$ and that t^α_{ij} is of left J-type $\pi^{\alpha(j)}$. Similarly, $V_{t^\alpha_{ij}}$ is spanned by the elements t^α_{kj} with $l_i \leq k \leq r_i$ and t^α_{ij} is of right J-type ${}^t\pi^{\alpha(i)}$. Since the $\{t^\alpha_{ij}\}_{\alpha \in \hat{\mathcal{A}}/1 \leq i,j \leq d_\alpha}$ constitute a linear basis of $\mathcal{A}$, this shows the existence of the decomposition. As for the uniqueness, suppose that $\sum_{j=1}^N a_j = 0$ where the a_j are non-zero and all of distinct left J-type. Since $\sum_j {}_{a_j}V$ is then a direct sum and since $a_j \in {}_{a_j}V$ for all j, we get a contradiction with $\sum_{j=1}^N a_j = 0$. This yields the uniqueness of the decomposition in left J-types. A similar argument applies to the decomposition in right J-types, thereby proving the lemma. □

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{U}$ be as before, and $J \subset \mathcal{U}$ be a $*$-invariant two-sided coideal. Suppose there exist a bijective, linear and anti-comultiplicative map $\sigma : \mathcal{A} \to \mathcal{A}$ and a bijective, linear map $\sigma_J : J \to J$, such that $\langle X, \sigma(a) \rangle = \langle \sigma_J(X), a \rangle$ for all $X \in J$, $a \in \mathcal{A}$.*

Now, if a given $a \in \mathcal{A}$ is of double J-type (π^1, π^2) then $\sigma(a)$ is of double J-type $(\pi^2 \circ \sigma_J, \pi^1 \circ \sigma_J)$.

Proof. If $\pi : J \to \mathrm{End}(V)$ is an irreducible representation of J, then so is $\pi \circ \sigma_J$. Suppose $a \in \mathcal{A}$ is of double J-type (π^1, π^2). Since

$$\begin{aligned} X.\sigma(a) &= (id \otimes X)\Delta\sigma(a) \\ &= (X \otimes id)(\sigma \otimes \sigma)\Delta(a) \\ &= \sigma(\sigma_J(X) \otimes id)\Delta(a) \\ &= \sigma(a.\sigma_J(X)) \end{aligned}$$

and similarly

$$\sigma(a).X = \sigma(\sigma_J(X).a),$$

this means that, as representations of J, we have $L_{\sigma(a)} = R_a \circ \sigma_J$ and $R_{\sigma(a)} = L_a \circ \sigma_J$. From this we see that $L_{\sigma(a)} = n(\pi^2 \circ \sigma_J)$ and $R_{\sigma(a)} = m(\pi^1 \circ \sigma_J)$, which proves the lemma. □

Since $\varepsilon(J) = 0$, we denote the trivial $*$-representation $\varepsilon : J \to \mathbb{C}$ by 0. Then we get from Lemma 2.2:

Corollary 2.3. *Under the same assumptions as in Lemma* 2.2, *we have that* $a \in \mathcal{A}$ *is of double J-type* $(0,0)$ *iff* $\sigma(a)$ *is of double J-type* $(0,0)$.

We are now able to state:

Theorem 2.4. *Suppose that the Hopf* $*$*-algebra* $\mathcal{A}$, *associated with a compact matrix quantum group, is in duality with the Hopf* $*$*-algebra* $\mathcal{U}$ *via the bilinear mapping* $\langle .,. \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$, *and let a* $*$*-invariant two-sided coideal J in* $\mathcal{U}$ *be given. Suppose furthermore that there exist a bijective, linear and anti-comultiplicative map* $\sigma : \mathcal{A} \to \mathcal{A}$ *and a bijective, linear map* $\sigma_J : J \to J$ *such that* $\langle X, \sigma(a) \rangle = \langle \sigma_J(X), a \rangle$ *for all* $X \in J$, $a \in \mathcal{A}$, *and such that* $\sigma(a) = a$ *for all* $a \in {}_J\mathcal{A}_J$.

Then $\dim(\mathrm{span}\{t_{ij}^\alpha\} \cap {}_J\mathcal{A}_J) \leq 1$ *for all irreducible unitary matrix corepresentations* (t_{ij}^α) *of* $\mathcal{A}$.

Proof. Using Lemma 2.1 we see that there is a well-defined projection

$$P : \mathcal{A} \to {}_J\mathcal{A}_J$$
$$a = \sum_{\substack{(\pi^1, \pi^2) \\ \text{finite}}} a_{\pi^1, \pi^2} \mapsto a_{0,0}.$$

By Corollary 2.3 this projection satisfies

$$P \circ \sigma = \sigma \circ P = P.$$

For a given irreducible unitary matrix corepresentation $(t_{ij}^\alpha)_{1 \leq i,j \leq n}$ of $\mathcal{A}$, suppose that $\mathrm{span}\{t_{ij}^\alpha\}_{1 \leq i,j \leq n} \cap {}_J\mathcal{A}_J = \mathrm{span}\{t_{ij}^\alpha\}_{1 \leq i,j \leq m}$, with $m \geq 1$. We have to prove that $m = 1$.
For $1 \leq i, j \leq m$, write out the equality

$$(id \otimes (\varepsilon \circ P) \otimes id)(\Delta \otimes id)\Delta(t_{ij}^\alpha) = (id \otimes (\varepsilon \circ P) \otimes id)(\Delta \otimes id)\Delta\sigma(t_{ij}^\alpha)$$

where $\varepsilon : \mathcal{A} \to \mathbb{C}$ is the counit mapping of $\mathcal{A}$. For the left-hand side we get

$$\sum_{k,l=1}^{n} t_{ik}^\alpha \otimes (\varepsilon \circ P)(t_{kl}^\alpha) \otimes t_{lj}^\alpha = \sum_{k=1}^{m} t_{ik}^\alpha \otimes t_{kj}^\alpha$$

and for the right-hand side, by the anti-comultiplicativity of σ,

$$\sum_{k,l=1}^{n} \sigma(t_{lj}^\alpha) \otimes (\varepsilon \circ P \circ \sigma)(t_{kl}^\alpha) \otimes \sigma(t_{ik}^\alpha) = \sum_{k=1}^{m} \sigma(t_{kj}^\alpha) \otimes \sigma(t_{ik}^\alpha)$$
$$= \sum_{k=1}^{m} t_{kj}^\alpha \otimes t_{ik}^\alpha$$

since $(\varepsilon \circ P \circ \sigma)(t_{kl}^\alpha) = (\varepsilon \circ P)(t_{kl}^\alpha) \neq 0$ iff $1 \leq k = l \leq m$, and in that case $(\varepsilon \circ P)(t_{kl}^\alpha) = 1$.
So we get that

$$\sum_{k=1}^{m} t_{ik}^\alpha \otimes t_{kj}^\alpha = \sum_{k=1}^{m} t_{kj}^\alpha \otimes t_{ik}^\alpha$$

which yields that $m \leq 1$ by the linear independence of the elements $\{t^\alpha_{rs} \otimes t^\alpha_{uv}\}$. This proves the theorem. $\square$

Remark 2.5. The previous setting can be slightly altered. Suppose that all the conditions of Theorem 2.4 are met, except that J is now no longer $*$-invariant. In addition, suppose that there exist Hopf algebra automorphisms $\eta : \mathcal{A} \to \mathcal{A}$ and $\eta' : \mathcal{U} \to \mathcal{U}$ such that $\langle \eta'(X), a\rangle = \langle X, \eta(a)\rangle$ for all $X \in \mathcal{U}$, $a \in \mathcal{A}$ and such that $(\eta'(J))^* = \eta'(J)$. Then we can apply Theorem 2.4 with J replaced by $\eta'(J)$, σ replaced by $\eta^{-1} \circ \sigma \circ \eta$ and σ_J by $\eta' \circ \sigma_J \circ (\eta')^{-1}$. This gives us that dim $(\mathrm{span}\{t^\alpha_{ij}\} \cap {}_{\eta'(J)}\mathcal{A}_{\eta'(J)}) \leq 1$ for all irreducible unitary matrix corepresentations (t^α_{ij}) of $\mathcal{A}$. Using the equality ${}_{\eta'(J)}\mathcal{A}_{\eta'(J)} = \eta^{-1}({}_J\mathcal{A}_J)$ and the fact that if (t^α_{ij}) runs through the equivalence classes of all irreducible unitary matrix corepresentations of $\mathcal{A}$ then so does $(\eta(t^\alpha_{ij}))$, we obtain that $\dim(\mathrm{span}\{t^\alpha_{ij}\} \cap {}_J\mathcal{A}_J) \leq 1$ for all irreducible unitary matrix corepresentations (t^α_{ij}) of $\mathcal{A}$.

Let us give two examples.

Example 1. Consider the Hopf $*$-algebras $\mathcal{A} = SU_q(2)$ and $\mathcal{U} = U_q(\mathfrak{su}(2))$ with generators α, β, γ, δ and A, B, C, D respectively (for the Hopf $*$-structures, see [K2]). They are in duality by

$$\left\langle A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$

$$\left\langle B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left\langle C, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left\langle D, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}.$$

For all $\tau \in \mathbb{R}$, define the element $X_\tau \in \mathcal{U}$ by

$$X_\tau = iq^{1/2}B - iq^{-1/2}C - \frac{q^\tau - q^{-\tau}}{q - q^{-1}}(A - D)$$

as in [K2], and put $J_\tau = \mathrm{span}\{X_\tau\}$. Furthermore, define the multiplicative mappings

$$\sigma : \mathcal{A} \to \mathcal{A}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} \alpha & -q\gamma \\ -q^{-1}\beta & \delta \end{pmatrix}$$

and

$$\sigma_{J_\tau} = id : J_\tau \to J_\tau$$

which satisfy the equality $\langle X, \sigma(a)\rangle = \langle \sigma_{J_\tau}(X), a\rangle$ for all $a \in \mathcal{A}$, $X \in J_\tau$. From (4.8) in [K2] we know that the algebra ${}_{J_\tau}\mathcal{A}_{J_\tau}$ is equal to the polynomial algebra in the element ρ_τ, where ρ_τ is given by

$$\rho_\tau = \alpha^2 + q^{-1}\beta^2 + q\gamma^2 + \delta^2 + i(q^{-\tau} - q^\tau)(q\delta\gamma - q\gamma\alpha + \beta\alpha - \delta\beta) + (q^\tau - q^{-\tau})^2\beta\gamma$$

(the element $a_\tau = \rho_\tau + ((q^\tau - q^{-\tau})^2/q + q^{-1})1_{\mathcal{A}}$ is the unique biinvariant element, up to constants, in the linear subspace $\mathcal{A}^1 = \operatorname{span}\{t^1_{ij} : i,j = -1,0,1\}$ of $\mathcal{A}$). So we see that the restriction of σ to ${}_{J_\tau}\mathcal{A}_{J_\tau}$ is the identity mapping.

Finally, let η and η' be the Hopf algebra automorphisms given on the generators by

$$\eta : \mathcal{A} \to \mathcal{A}$$
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & q^{-1/2}\beta \\ q^{1/2}\gamma & \delta \end{pmatrix}$$

and

$$\eta' : \mathcal{U} \to \mathcal{U}$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & q^{-1/2}B \\ q^{1/2}C & D \end{pmatrix}.$$

They satisfy $\langle X, \eta(a)\rangle = \langle \eta'(X), a\rangle$ for all $a \in \mathcal{A}$, $X \in \mathcal{U}$. It immediately follows that

$$\eta'(J_\tau) = \operatorname{span}\left\{ i(B - C) - \frac{q^\tau - q^{-\tau}}{q - q^{-1}}(A - D) \right\}$$

is a $*$-invariant two-sided coideal in $\mathcal{U}$. Thus, by Remark 2.5, the statement of Theorem 2.4 holds.

Example 2. In this last example we adopt the framework and notation of Noumi [N2]. Throughout we will fix $q \in (0,1)$.

Consider the Hopf $*$-algebras $\mathcal{A} = A(U_q(N))$ and $\mathcal{U} = U_q(\mathfrak{gl}(N))$. They have generators t_{ij} $(1 \le i,j \le N)$, $det_q T^{-1} = det_q^{-1}$, respectively q^λ $(\lambda \in P^*)$, e_k, f_k $(1 \le k \le N-1)$. Here P denotes the weight lattice $P = \sum_{i=1}^N \mathbb{Z}\varepsilon_i$, which is identified with P^* via the bilinear form $\langle .,. \rangle : P \times P \to \mathbb{Z}$, $\langle \varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$. For the Hopf structures on $\mathcal{A}$ and $\mathcal{U}$ we refer to [N2], Sections 1.1, 1.2. The Hopf structure for $\mathcal{A}$ is the same as in [NYM] (except for slightly changed notation), and we used it already in Example 2 of Section 1. However, the Hopf structure imposed on $\mathcal{U}$ differs from the one in [NYM], but there is a Hopf algebra isomorphism between these two sending q^λ, e_i, f_i of [NYM] to $q^\lambda, q^{h_i/2}e_i =: e_i, f_i\, q^{-h_i/2} =: f_i$ of [N2] (here $h_i = \varepsilon_i - \varepsilon_{i+1}$).

In $\mathcal{U}$ there are singled out the so-called L-operators $L^\pm_{ij}$ $(1 \le i,j \le N)$ of [RTF]. They are given by means of a unique family of elements E_{ij} $(1 \le i,j \le N$, $i \ne j)$ in $\mathcal{U}$ that satisfy

$$E_{i,i+1} = e_i$$
$$E_{ij} = E_{ik}E_{kj} - qE_{kj}E_{ik} \quad (i < k < j)$$
$$E_{i+1,i} = f_i$$
$$E_{ij} = E_{ik}E_{kj} - q^{-1}E_{kj}E_{ik} \quad (i > k > j),$$

and have the form

$$L_{ii}^{+} = q^{\varepsilon_i}$$

$$L_{ij}^{+} = (q - q^{-1})q^{\varepsilon_i}E_{ji} \quad (i < j)$$

$$L_{ij}^{+} = 0 \ (i > j)$$

$$L_{ii}^{-} = q^{-\varepsilon_i}$$

$$L_{ij}^{-} = -(q - q^{-1})E_{ji}q^{-\varepsilon_j} \quad (i > j)$$

$$L_{ij}^{-} = 0 \ (i < j).$$

On the respective algebras $\mathcal{A}$ and $\mathcal{U}$ we take the following $*$-operations:

$$t_{ij}^{*} = S(t_{ji})$$

$$(det_q^{-1})^{*} = det_q$$

and

$$(L_{ij}^{\pm})^{*} = S(L_{ji}^{\mp}).$$

On the Chevalley generators q^{λ}, e_i, f_i this involution reads

$$(q^{\lambda})^{*} = q^{\lambda}$$

$$e_i^{*} = q^{-1}f_i q^{\varepsilon_i - \varepsilon_{i-1}}$$

$$f_i^{*} = qq^{-\varepsilon_i + \varepsilon_{i+1}}e_i.$$

Then $\mathcal{A}$ and $\mathcal{U}$ become Hopf $*$-algebras. They are in duality if we let

$$\langle L_{ij}^{\pm}, t_{kl}\rangle = R_{ik;jl}^{\mp}$$

$$\langle L_{ij}^{\pm}, det_q^{-1}\rangle = q^{\mp 1}\delta_{ij},$$

where

$$R^{+} = \sum_{1 \le i,j \le N} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{1 \le i < j \le N} e_{ij} \otimes e_{ji}$$

$$R^{-} = \sum_{1 \le i,j \le N} q^{-\delta_{ij}} e_{ii} \otimes e_{jj} - (q - q^{-1}) \sum_{1 \le j < i \le N} e_{ij} \otimes e_{ji}$$

are R-matrices, satisfying the Yang–Baxter equation, that correspond to root system A_n. Then it follows that on the Chevalley generators the pairing is given by

$$\langle q^{\lambda}, t_{kl}\rangle = \delta_{kl}\, q^{\langle\lambda, \varepsilon_k\rangle}$$

$$\langle e_i, t_{kl}\rangle = \delta_{ik}\,\delta_{i+1,l}$$

$$\langle f_i, t_{kl}\rangle = \delta_{i+1,k}\,\delta_{il}.$$

As in [N2] we will consider biinvariance with respect to deformations of the Lie algebras of $\mathfrak{so}(n)$ $(N = n)$ and $\mathfrak{sp}(2n)$ $(N = 2n)$, and refer to them as cases (SO) and (Sp) respectively. In Noumi's setting the deformation of these Lie algebras involve an invertible scalar matrix $J(a)$ which depends on an n-dimensional vector with nonzero scalar coordinates. However, in what follows we will take $a = 1 := (1, \ldots, 1)$. This means that $J(a) = J(1)$ takes the form

(SO) $\quad J(1) = Id_n$

(Sp) $\quad J(1) = \sum_{k=1}^{n} (e_{2k-1,2k} - q e_{2k,2k-1}).$

With these matrices we define the following matrix M with entries in $\mathcal{U}$:

$$M = L^+ - J(1) S(L^-)^t J(1)^{-1},$$

where t stands for matrix transposition. The two-sided coideal in $\mathcal{U}$ that we will consider will then be

$$J = \sum_{i,j=1}^{N} \mathbb{C} M_{ij}.$$

It is easily seen that

$$M_{ij}^* = -(J(1) M J(1)^{-1})_{ij},$$

whence $J^* = J$ (see [N2], (3.31)). Furthermore, the biinvariants in $\mathcal{A}$ form the commutative algebra

$$_J\mathcal{A}_J = \mathbb{C}[w_1, \ldots, w_{n-1}, det_q^{\pm 1}]$$

in which the w_r $(1 \leq r \leq n-1)$ are given as

(SO) $\quad w_r = \sum_{\substack{1 \leq i_1 < \cdots < i_r \leq n \\ 1 \leq j_1 < \cdots < j_r \leq n}} (\xi_{j_1 \cdots j_r}^{i_1 \cdots i_r})^2$

(Sp) $\quad w_r = \sum_{\substack{1 \leq i_1 < \cdots < i_r \leq n \\ 1 \leq j_1 < \cdots < j_r \leq n}} \xi_{2j_1-1, 2j_1, \ldots, 2j_r-1, 2j_r}^{2i_1-1, 2i_1, \ldots, 2i_r-1, 2i_r}$

(see [N2], Lemma 4.13, (4.36), (4.41) and Theorem 4.7). The element $\xi_{j_1 \cdots j_r}^{i_1 \cdots i_r}$ denotes the quantum minor determinant in $\mathcal{A}$ corresponding to the r-tuples $\{1 \leq i_1 < \cdots < i_r \leq n\}$ and $\{1 \leq j_1 < \cdots < j_r \leq n\}$ (cf. [N2], (4.34)).
If we now define the following mappings on $\mathcal{A}$ and J

$$\sigma : \mathcal{A} \to \mathcal{A}$$

$$t_{ij} \mapsto t_{ji}$$

$$det_q^{-1} \mapsto det_q^{-1}$$

extended as an algebra homomorphism, i.e. $\sigma = * \circ S$ on the generators of $\mathcal{A}$, and

$$\sigma_J(M_{ij}) = M_{ij}^*,$$

extended linearly to J, then it is readily seen that all the conditions of Theorem 2.4 are satisfied (note that $\sigma(w_r) = w_r$ by the remark after (1.12) in [NYM]). By suitable rewriting it can be seen that the case $\tau = 0$ of Example 1 of Section 2 is the special case $n = 2$, $J(1)$ of type (SO) in the present example.

Remark 2.6. In their paper [CV] Chapovsky and Vainerman gave a definition of a Gel'fand pair of Hopf algebras. Their definition is as follows. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two Hopf algebras over the complex numbers, such that there exist a surjective Hopf algebra homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ and an invariant functional $\nu : \mathcal{B} \to \mathbb{C}$. Then the pair $(\mathcal{A}, \mathcal{B})$ is called a Gel'fand pair if the mapping $\tilde{\Delta} := (id \otimes \nu \circ \psi \otimes id)(\Delta \otimes id)\Delta$, which is a coproduct on ${}_{\mathcal{B}}\mathcal{A}_{\mathcal{B}}$ (cf. [CV]), is cocommutative. It is shown in [CV] that this definition is equivalent to the one given in [K3], and our Section 1, if $\mathcal{A}$ and $\mathcal{B}$ are compact matrix quantum groups. It is immediately seen that in their more general setup the fulfilment of the conditions in our Theorem 1.2 will provide the cocommutativity of the coproduct $\tilde{\Delta}$, if in addition we assume that $\sigma_{\mathcal{B}}$ preserves the invariant integral ν (cf. proof of Theorem 1.2).

In [V] Vainerman gave a similar definition in the infinitesimal case. It again comes down to the cocommutativity, under a specific assumption, of a certain coproduct $\tilde{\Delta}$, now on the subalgebra ${}_{J_1}\mathcal{A}_{J_2}$ of $\mathcal{A}$. It should be noted however that the coideals J_1 and J_2 in his case are not necessarily the same, and also not necessarily $*$-invariant. Hence our Section 2 is a specialization of Vainerman's setup. It is immediate that in this special case the conditions of Theorem 2.4 imply the cocommutativity of $\tilde{\Delta}$ (which is essentially $(id \otimes (\varepsilon \circ P) \otimes id)(\Delta \otimes id)\Delta$ as in the proof of Theorem 2.4).

Remark 2.7. The examples given in this paper are all of Hopf $*$-algebras associated with a compact matrix quantum group (CMQG-algebras in the terminology of [K5]). However, we essentially only used that $\mathcal{A}$ is a so-called CQG-algebra, i.e. is associated with a (general) compact quantum group (see [K5] for the details). In other words, all results presented here hold equally well under the assumption that the Hopf $*$-algebra $\mathcal{A}$ is a CQG-algebra.

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